Generators and subgroups for Aut(F_3)

1995 J. Phys. A: Math. Gen. 28 379

(http://iopscience.iop.org/0305-4470/28/2/014)

J. Phys. A: Math. Gen. 28 (1995) 379–389. Printed in the UK

Generators and subgroups for $Aut(F_3)$

Peter Kramer

Institut für Theoretische Physik der Universität, Tübingen, Germany

Received 19 September 1994, in final form 11 November 1994

Abstract. The automorphisms of the free group F_3 with three generators form the group $Aut(F_3)$. New involutive generators and relations for $Aut(F_3)$ are given which explain the relations given by Nielsen. Finite Coxeter groups and (normal) subgroups are derived which admit generalizations to $n \geqslant 3$.

1. Introduction

The free group F_n with n generators provides a description of non-commutative systems in physics. Therefore the group $Aut(F_n)$ of its automorphisms, studied first by Nielsen [10], is of considerable interest for these systems. The groups F_n and $Aut(F_n)$ have been used to formulate a non-commutative crystallography which encompasses aspects of quasicrystals, [1–3, 5–7]. In [8] a new system of involutive generators and relations for $Aut(F_2)$ was introduced to derive the relations due to Nielsen. In what follows we present a similar analysis for $Aut(F_3)$. From Nielsen's analysis, $Aut(F_3)$ shows almost the full complexity of $Aut(F_n)$. Involutive generators and new relations are given and used to derive and explain the relations of Nielsen. Certain (normal) subgroups, among them the finite Coxeter groups A_3 and B_3, are explicitly constructed, with counterparts for $n \geqslant 3$.

2. Nielsen generators and relations

Let F_n denote the free group with n generators, and denote by $\Phi_n := Aut(F_n)$ its group of automorphism [10]. We present a set of four generators and relations R_i for Φ_n due to Nielsen. The sign $a \rightleftharpoons b$ indicates that a, b commute.

Proposition 1 (Nielsen 1924) [10, 9]. $Aut(F_n)$ is the group

$$\Phi_n := \langle P, Q, \sigma, U | R_1 \ldots R_{18} \rangle \tag{1}$$

with relations

$$\begin{aligned}
\sigma^2 &= P^2 = e \qquad (R_1, R_2) \\
(P\sigma PU)^2 &= e \qquad (R_3) \\
U^{-1}PUP\sigma U\sigma P\sigma &= e \qquad (R_4) \\
U &\rightleftharpoons \sigma U \sigma \qquad (R_5) \\
(QP)^{n-1} &= Q^n = e \qquad (R_6, R_7) \\
P &\rightleftharpoons Q^{-i}PQ^i \qquad i = 1 \ldots [n/2] \qquad (R_8) \\
\sigma &\rightleftharpoons Q^{-1}PQ \qquad (R_9)
\end{aligned} \tag{2a}$$

$$
\begin{aligned}
\sigma &\rightleftharpoons QP \qquad (R_{10}) \\
\sigma &\rightleftharpoons Q^{-1}\sigma Q \qquad (R_{11}) \\
(PQ^{-1}UQP)(Q^{-1}UQ) &= UQ^{-1}UQU^{-1} \qquad (R_{12}) \\
U &\rightleftharpoons Q^{-2}PQ^2 \qquad (R_{13}) \\
U &\rightleftharpoons QPQ^{-1}PQ \qquad (R_{14}) \\
U &\rightleftharpoons Q^{-2}\sigma Q^2 \qquad (R_{15}) \\
U &\rightleftharpoons Q^{-2}UQ^2 \qquad (R_{16}) \\
U &\rightleftharpoons PQ^{-1}\sigma U\sigma QP \qquad (R_{17}) \\
U &\rightleftharpoons PQ^{-1}PQPUPQ^{-1}PQP \qquad (R_{18}).
\end{aligned}
\tag{2b}
$$

The generators are defined as automorphisms of $F_n = \langle x_1, \ldots x_n \rangle$ by

$$
\begin{array}{llllllll}
e: & x_1 & x_2 & x_3 & \ldots & x_{n-1} & x_n \\
P: & x_2 & x_1 & x_3 & \ldots & x_{n-1} & x_n \\
Q: & x_2 & x_3 & x_4 & \ldots & x_n & x_1 \\
\sigma: & (x_1)^{-1} & x_2 & x_3 & \ldots & x_{n-1} & x_n \\
U: & x_1x_2 & x_2 & x_3 & \ldots & x_{n-1} & x_n.
\end{array}
\tag{3}
$$

We now give the modifications of the relations for the case $n = 3$. From $Q^3 = e$, R_7, one finds that relations R_9 and R_{10} become equivalent and that R_{14} is trivial. With the help of (3) one finds that R_{13} and R_{16} are not valid and must be dropped for Φ_3.

We have arranged the relations in an order such that the group $\Phi_2 := Aut(F_2)$ which acts only on $\langle x_1, x_2 \rangle$ is given by

$$\Phi_2 := \langle P, U, \sigma \,|\, R_1 \ldots R_5 \rangle. \tag{4}$$

3. New generators and relations

In this section we define a group in terms of generators and relations, which in section 4 is shown to be isomorphic to the group $\Phi_3 = Aut(F_3)$ of the free group F_3.

Definition 2. The group Φ_3' is defined in terms of five generators and 20 relations Q_i as given below:

$$\Phi_3' := \langle c_{12}, c_{23}, c_{34}, \sigma_1, c_2 \,|\, Q_1 \ldots Q_{20} \rangle. \tag{5}$$

We shall arrange the relations Q_i into sets which determine certain subgroups of Φ_3'.

The first set of relations determines a Coxeter group A_3:

$$
\begin{aligned}
&A_3 : \langle c_{12},\ c_{23},\ c_{34} \rangle \\
&\quad (c_{ii+1})^2 = e \qquad i = 1, 2, 3 \qquad (Q_1, Q_2, Q_3) \\
&\quad (c_{12}c_{23})^3 = e \qquad (c_{23}c_{34})^3 = e \qquad (Q_4, Q_5) \\
&\quad c_{12} \rightleftharpoons c_{34} \qquad (Q_6).
\end{aligned}
\tag{6}
$$

In the standard notation for Coxeter groups [4], each generator carries a single index. Here we use the isomorphism $A_n \sim S_{n+1}$ and interprete the generators of A_3 as transpositions with respect to the pair of indices $(i, i+1)$. Moreover we shall denote a general transposition of the pair of indices (i, j) by $c_{ij} = c_{ji} \in A_3$. The second set of relations determines a Coxeter group B_3:

$$
\begin{aligned}
B_3 := \langle c_{34},\ c_{13},\ \sigma_1 \rangle \\
&(c_{34})^2 = (c_{13})^2 = e \\
&(\sigma_1)^2 = e \qquad (Q_7) \\
&(c_{34}c_{13})^3 = e \\
&(c_{13}\sigma_1)^4 = e \qquad (Q_8) \\
&c_{34} \rightleftharpoons \sigma_1 \qquad (Q_9) \\
&c_{13} := c_{12}c_{23}c_{12}.
\end{aligned} \tag{7}
$$

Note that $A_3 \cap B_3 = A_2$ and that part of the equations arise already from $Q_1 \dots Q_6$.

The third set of relations is connected with subgroups of Φ_3' described in definition 3 and in section 5.

$$\sigma_1 \rightleftharpoons c_{23}(\sigma_1 c_{13})^2 c_{23} \qquad (Q_{10}) \tag{8}$$

$$\sigma_1 \rightleftharpoons c_{14}c_{23}\sigma_1 c_{23}c_{14} \qquad (Q_{11}). \tag{9}$$

Definition 3. We define a subgroup $\Psi_4 < \Psi_3'$ by

$$\Psi_4 := \langle c_{12}, c_{23}, c_{34}, \sigma_1 | \ Q_1 \dots Q_{11} \rangle. \tag{10}$$

This subgroup has only the first four generators of Φ_3' and all the relations between them.

Now we turn to subgroups of Φ_3' isomorphic to Φ_2.

Proposition 4. The subgroup of Φ_3' generated by $\langle c_{23}, c_{13}, \sigma_1 \rangle$ fulfills the relations

$$
\begin{aligned}
(c_{23})^2 &= (c_{13})^2 = (\sigma_1)^2 = e \\
(c_{23}c_{13})^3 &= e \\
(c_{13}\sigma_1)^4 &= e \qquad (Q_8) \\
\sigma_1 &\rightleftharpoons c_{23}(c_{13}\sigma_1)^2 c_{23} \qquad (Q_{10}).
\end{aligned} \tag{11}
$$

and is a subgroup of Ψ_4 isomorphic to Φ_2.

Proof. The relations of (11) follow from $Q_1 \dots Q_5, Q_7$. It was shown in [8] that a group with three generators and the relations given in (11) is isomorphic to Φ_2. Clearly the three generators $\langle c_{23}, c_{13}, \sigma_1 \rangle$ generate a subgroup of Ψ_4 given in definition 3. □

We turn to relations involving the generator c_2. It is an involution, hence we require

$$(c_2)^2 = e \qquad (Q_{12}). \tag{12}$$

The next relations are given in terms of the elements

$$X_1 := c_2 c_3 c_{13} c_2 \qquad X_2 := c_2 c_{23} \qquad c_3 := c_{12} c_2 c_{12}. \tag{13}$$

For the two elements X_1, X_2 we require as the fourth set of relations the following transformations under conjugation $(g, X) \to X^g := gXg^{-1}$ with $\langle c_2, c_3, \sigma_1 \rangle$:

$$
\begin{aligned}
&(X_1)^{c_2} = X_1 X_2 \qquad &&(X_2)^{c_2} = (X_2)^{-1} \qquad &&(Q_{13}, Q_{14}) \\
&(X_1)^{c_3} = (X_2)^{-1} \qquad &&(X_2)^{c_3} = (X_1)^{-1} \qquad &&(Q_{15}, Q_{16}) \\
&(X_1)^{\sigma_1} = (X_1)^{-1} \qquad &&(X_2)^{\sigma_1} = X_2 \qquad &&(Q_{17}, Q_{18}).
\end{aligned} \tag{14}
$$

The notation X_1, X_2 will be explained later.

Proposition 5. The subgroup of Φ_3' generated by $\langle c_2, c_3, \sigma_1\rangle$, with relations obtained from (11) by the replacement

$$c_{23} \to c_2 \qquad c_{13} \to c_3 \qquad \sigma_1 \to \sigma_1 \tag{15}$$

is isomorphic to Φ_2.

Proof. The proof involves a transformation from the generators given in proposition 4 and uses the relations of (14). From (13) we find

$$c_{23} = c_2 X_2 \qquad c_{13} = c_3 X_1 X_2. \tag{16}$$

Inserting these expressions into the relations of (11) and using the conjugation properties (14) of X_1, X_2 in each case, one finds that all these relations are valid under the replacement (15). As an example we consider

$$\begin{aligned} e &= (c_{23}c_{13})^3 \\ &= (c_2X_2c_3X_1X_2)^3 \\ &= (c_2c_3X_2)^3 \\ &= (c_2c_3)^3(c_2c_3)^{-2}X_2(c_2c_3)^2(c_2c_3)^{-1}X_2(c_2c_3)X_2 \\ &= (c_2c_3)^3(X_2^{-1}X_1^{-1})X_1X_2 \\ &= (c_2c_3)^3. \end{aligned} \tag{17}$$

All the relations together determine a group Φ_2. □

The two subgroups of propositions 4, 5 are isomorphic but not conjugate to one another. For the conjugation of the generator σ_1 with elements from A_3 we introduce the notation $\sigma_{212} := \sigma_1$. In view of the stability (equation (7)) Q_9 of σ_{212} under c_{34}, there are altogether 12 conjugates of σ_1 under A_3 which we denote by

$$i \neq j : \sigma_{iji} := (\sigma_{212})^{c_{j1}c_{i2}}. \tag{18}$$

Finally we require two more relations between c_2 and generators not in Φ_2:

$$\begin{aligned} c_2 &\rightleftharpoons \sigma_{343} = (\sigma_1)^{c_{41}c_{32}} \qquad (Q_{19}) \\ c_{23} &= c_2\sigma_2 q \qquad (Q_{20}) \\ q &:= c_{23}c_{14}c_2c_{14}c_{23} \\ \sigma_2 &:= c_3\sigma_1 c_3. \end{aligned} \tag{19}$$

4. Isomorphism with $Aut(F_3)$

In this section we relate Φ_3' to $\Phi_3 = Aut(F_3)$. First we relate the sets of generators, then we derive the Nielsen relations of section 2 from the relations $Q_1 \ldots Q_{20}$ and thus prove the isomorphism of Φ_3' with $Aut(F_3)$. The proofs are independent of the action of $Aut(F_3)$ on F_3.

The group Φ_3' contains two distinct subgroups isomorphic to Φ_2 (compare proposition 4, 5). Nielsen's Φ_2 is identified by

Lemma 6. Upon setting

$$\sigma = \sigma_1 \qquad P = \sigma_1 c_3 \sigma_1 \qquad U = c_3\sigma_1 c_3 c_2 \qquad c_3 := c_2 c_{12} c_2 \tag{20}$$

and using the relations $Q_1 \ldots Q_{18}$, the Nielsen relations $R_1 \ldots R_5$ are fulfilled, and the subgroup $\Phi_2 = \langle c_2, c_3, \sigma_1\rangle$ is generated.

Proof. The lemma is obtained from proposition 5 derived from $Q_{13} \dots Q_{18}$ and from the transformations (20). The Nielsen relations then follow as given in [8], proposition 1. □

Lemma 7. Define the Coxeter group conjugate to B_3 with generators

$$P_{23} := (c_{34})^{c_2 c_{23} \sigma_1} \qquad P_{12} := (c_{13})^{c_2 c_{23} \sigma_1} \qquad \sigma_1 = (\sigma_1)^{c_2 c_{23} \sigma_1}. \tag{21}$$

Then with the transformations

$$P := P_{12} \qquad Q := P_{23} P_{12} \qquad \sigma = \sigma_1 \tag{22}$$

the Nielsen relations $R_1, R_2, R_6 \dots R_{11}$ are fulfilled.

Proof. Clearly a conjugation transformation preserves the commutator properties of the group B_3. Next we note that the expressions for P given in (20) and (22) agree by use of Q_{18}. The Nielsen relations $R_1, R_2, R_6 \dots R_{11}$ involve only the generators P, Q, σ. It is easy to see that they arise from the Coxeter group B_3, hence from $Q_1 \dots Q_{11}$ and equations (21), (22), and yield this group in Nielsen form. □

The two-index notation for B_3 introduced in lemma 7 anticipates the action of this group on the generators of F_3, but it has no simple relation with the two-index notation for A_3 introduced in (6). For later use we note that from (13), (14), Q_{18} we have

$$c_2 c_{23} \sigma_1 = \sigma_1 c_2 c_{23} \tag{23}$$

which allows us to rewrite equation (21) in the form

$$P_{23} = (c_{24})^{\sigma_1 c_2} \qquad P_{12} = (c_{12})^{\sigma_1 c_2} \qquad \sigma_1 = (\sigma_{313})^{\sigma_1 c_2}. \tag{24}$$

Taking the group B_3 in Nielsen form and using equations (19), (23) one finds

$$\sigma_2 = P_{12} \sigma_1 P_{12} \qquad \sigma_3 := P_{23} \sigma_2 P_{23}. \tag{25}$$

The elements σ_i are easily shown to commute with one another. They are permuted under conjugation with the permutations of B_3.

With the expressions of lemmas 6 and 7, the Nielsen generators are given as functions of the generators (5) of Φ_3'. We turn now to the inverse transformations.

Proposition 8. The generators of Φ_3' can be expressed as functions of the Nielsen generators.

Proof. From the subgroup Φ_2 of lemma 6 one finds (cf [8]) the expressions

$$\sigma_1 = \sigma \qquad c_2 = \sigma P \sigma P \sigma U \qquad c_3 = \sigma P \sigma. \tag{26}$$

We obtain by conjugation from (24)

$$c_{12} = c_2 \sigma P_{12} \sigma c_2 \qquad c_{24} = c_2 \sigma P_{23} \sigma c_2. \tag{27}$$

With $P_{12} = P$, $P_{23} = QP$ and (26), the right-hand sides of (27) become functions of the Nielsen generators. With $c_{14} = c_{12} c_{24} c_{12}$ we get this transposition as a function of Nielsen generators. From Q_{20}, (A1) and (29) one may now write the generator c_{23} as a function of the Nielsen generators. Since $c_{34} = c_{23} c_{14} c_{23}$, we can then express the five generators of Φ_3' as functions of the Nielsen generators. □

Proposition 9. The generators of Φ_3' act on the free group F_3 according to

$$\begin{array}{llll} e: & x_1 & x_2 & x_3 \\ c_{12}: & (x_1)^{-1} & x_1x_2 & x_3 \\ c_{23}: & x_1x_2 & (x_2)^{-1} & x_2x_3 \\ c_{34}: & x_1 & x_2x_3 & (x_3)^{-1} \\ \sigma_1: & (x_1)^{-1} & x_2 & x_3 \\ c_2: & x_1x_2 & (x_2)^{-1} & x_3. \end{array} \tag{28}$$

Proof. We use proposition 8 to pass to the Nielsen generators and equation (3) to find the action on F_3. □

Lemma 10. The Nielsen relations (2), $R_{12} \ldots R_{18}$ follow by transforming with proposition 8 to the generators of Φ_3' and use of the relations $Q_1 \ldots Q_{20}$.

Proof. See appendix A.

Proposition 11. The groups

$$\begin{aligned} \Phi_3 &= \langle P, Q, \sigma, U \mid R_1 \ldots R_{18} \rangle \\ \Phi_3' &= \langle c_{12}, c_{23}, c_{34}, \sigma_1, c_2 \mid Q_1 \ldots Q_{20} \rangle \end{aligned} \tag{29}$$

are isomorphic.

Proof. From lemma 10, it follows that $\Phi_3' \leqslant \Phi_3$. Through the expressions of proposition 9, *it can be checked that relations* $Q_1 \ldots Q_{20}$ *of* Φ_3' *hold true in terms of actions on* F_3. *It* follows that Φ_3' produces no relations beyond Φ_3, hence Φ_3' is isomorphic to Φ_3. □

5. Subgroups of $Aut(F_3)$

The generators for Φ_3 given in section 3 were already adapted to the subgroups A_3, B_3 and Φ_2. In the latter case we found two isomorphic but non-conjugate subgroups. Their relation may be interpreted by deriving the action of the group elements X_1, X_2 from (13) and (28). One finds

$$\begin{array}{llll} e: & x_1 & x_2 & x_3 \\ X_1: & x_1 & x_2 & x_1x_3 \\ X_2: & x_1 & x_2 & x_2x_3. \end{array} \tag{30}$$

It is easy to see that these two elements of Φ_3 generate a subgroup isomorphic to F_2. Considering now the action of Φ_2 by conjugation on X_1, X_2 given in (14), one finds that $F_2 = \langle X_1, X_2 \rangle$ may be combined with $\Phi_2 = \langle c_2, c_3, \sigma_1 \rangle$ into a semidirect product $F_2 \times_s \Phi_2$ *in which the factor* F_2 *is an invariant subgroup.*

A different type of subgroup is found within Ψ_4:

Proposition 12. In the group Ψ_4, the elements

$$\begin{array}{ll} S_1 := \sigma_{121}\sigma_{131}\sigma_{141} & S_2 := \sigma_{212}\sigma_{232}\sigma_{242} \\ S_3 := \sigma_{313}\sigma_{323}\sigma_{343} & S_4 := \sigma_{414}\sigma_{424}\sigma_{434} \end{array} \tag{31}$$

are involutive and generate a normal subgroup $\mathcal{X}_4$. The conjugation properties of the elements (31) under $p \in A_3$ and σ_1 are

$$(S_i)^p = S_{p(i)} \tag{32}$$

$$(S_1)^{\sigma_1} = S_2S_1S_2 \qquad (S_2)^{\sigma_1} = S_2 \qquad (S_3)^{\sigma_1} = S_3 \qquad (S_4)^{\sigma_1} = S_4. \tag{33}$$

Proof. From Q_8 one finds that $(c_{13}\sigma_1)^2 = (\sigma_1 c_{13})^2$ so that $\sigma_{212} \rightleftharpoons \sigma_{232}$. By conjugation with c_{34} and use of Q_9, one finds $\sigma_{212} \rightleftharpoons \sigma_{242}$ and, by conjugation with c_{14}, $\sigma_{242} \rightleftharpoons \sigma_{232}$. Hence S_2 is a product of three commuting involutions so that $(S_2)^2 = e$. Since S_1, S_3, S_4 can be obtained from S_2 by conjugation, they are also involutions. To examine the normal property it suffices to conjugate the S_i with the four generators of Ψ_4. Under A_3, one easily finds that S_1, S_2, S_3, S_4 are permuted as given in (35). Under σ_1 one finds with the help of Q_{10}, Q_{11} and their conjugates the results of (36). Since the conjugates under all the generators of Ψ_4 are expressed by S_1, S_2, S_3, S_4, the normal property follows. □

From this group we pass to the following proposition.

Proposition 13. The subgroup $\mathcal{F}_3 = \langle T_1, T_2, T_3 \rangle$

$$T_1 = S_1 S_2 \qquad T_2 = S_2 S_3 \qquad T_3 = S_3 S_4, \tag{34}$$

written in terms of the involutive generators of $\mathcal{X}_4$, is the normal subgroup $Inn(F_3) \lhd Aut(F_3)$ of inner automorphisms in Φ_3 and is isomorphic to F_3. Under conjugation with the generators of $Aut(F_3)$, the transformation laws are

$$\begin{array}{llll} T_i: & T_1 & T_2 & T_3 \\ (T_i)^{c_{12}}: & (T_1)^{-1} & T_1T_2 & T_3 \\ (T_i)^{c_{23}}: & T_1T_2 & (T_2)^{-1} & T_2T_3 \\ (T_i)^{c_{34}}: & T_1 & T_2T_3 & (T_3)^{-1} \\ (T_i)^{\sigma_1}: & (T_1)^{-1} & T_2 & T_3 \\ (T_i)^{c_2}: & T_1T_2 & (T_2)^{-1} & T_3 \end{array} \tag{35}$$

Moreover the generators $\langle T_1, T_2, T_3 \rangle$ are the images of $\langle x_1, x_2, x_3 \rangle$ under the isomorphism.

Proof. See appendix B.

Computation of the action of $\langle T_1, T_2, T_3 \rangle$ on F_3 shows that these automorphisms generate the inner automorphisms $T_i : w \to (w)^{x_i^{-1}}$, $i = 1, 2, 3$ respectively. For any free group, $Inn(F_n)$ is isomorphic to F_n. Comparing the action (35) of the generators by conjugation on $\langle T_1, T_2, T_3 \rangle$ with their action on $\langle x_1, x_2, x_3 \rangle$ given in (28), one finds the correspondence $T_i \sim x_i$ under this isomorphism.

6. Towards non-commutative crystallography

As one possible field of applications we indicate here the elements of non-commutative (NC) crystallography. The subgroups of $Aut(F_3)$ given in the previous sections allow one to construct generalized notions for the translation, point and space groups of standard crystallography.

From proposition 13, an isomorphic image of the free group F_3 appears in $Aut(F_3)$ as the normal subgroup $\mathcal{F}_3$. We interprete this normal subgroup as an NC *translation group* $\mathcal{T} := \mathcal{F}_3 = \langle T_1, T_2, T_3 \rangle$ with the following motivation:

(i) By the homomorphism $hom_1 : F_3 \to Z^3$ called abelianization, F_3 and hence $\mathcal{T}$ maps into the Abelian group Z^3 with three generators, isomorphic to a translation group.
(ii) By extending the affine representation given in [8] from $Aut(F_2)$ to $Aut(F_3)$, it can be shown that the generators $\langle T_1, T_2, T_3 \rangle$ admit a representation by discrete shifts on a 3D lattice. Any closed path formed from discrete shifts represents an element from the kernel $ker(hom_1)$.

There is a second homomorphism $hom_2 : Aut(F_3) \to Gl(3, Z)$ [9] which links automorphisms to symmetry transformations on a lattice. Moreover the two Coxeter subgroups given in section 3 (6), (7) are isomorphic to crystallographic point groups, $A_3 = T_d = \bar{4}3m$, $B_3 = O_h = m3m$. They may then be taken as *finite point groups* in NC crystallography.

Clearly the generators of these groups must map by conjugation $\mathcal{T} \to \mathcal{T}$. It is not difficult to check that the subgroup $\mathcal{T}$, combined with each one of these point groups, fulfills the condition for a semidirect product in which the subgroup $\mathcal{T}$ is normal. This allows one to introduce the two NC *symmorphic space groups* $\mathcal{T} \times_s A_3$ and $\mathcal{T} \times_s B_3$.

We have then within $Aut(F_3)$ all the elements of a NC crystallography in which the commutativity of the translation group is broken and the point symmetry can be preserved. Quasicrystals lack the periodicity of crystals and therefore display a type of order where this scheme may apply. To implement it one should explore within this scheme production rules that can describe quasicrystals (cf [2]).

7. Conclusion

We have given new involutive generators and relations for $Aut(F_3)$. The new generators for $Aut(F_3)$ are adapted to some finite and infinite subgroups. In all cases we give the explicit generators of these subgroups. The new relations lead to an interpretation of the relations given by Nielsen: they are closely connected to the existence of certain (normal) subgroups of $Aut(F_3)$.

We indicate the generalization to $n \geqslant 3$: the finite Coxeter groups A_n and B_n appear in full analogy as subgroups of $Aut(F_n)$. This is shown in [1] for A_n and from Nielsen [9, 10] clearly for B_n. Both these subgroups together generate a subgroup Ψ_{n+1} which generalizes the group Ψ_4 of definition 3. Their intersection is $A_n \cap B_n = A_{n-1}$. Ψ_{n+1} has a normal subgroup $\mathcal{X}_{n+1}$ which generalizes $\mathcal{X}_4$ of proposition 12, and $\mathcal{X}_{n+1}$ has a subgroup normal in $Aut(F_n)$ which is $Inn(F_n)$ and isomorphic to F_n. In $Aut(F_3)$, the new generator c_2 appeared which does not belong to Ψ_4. The case $n = 2$ is special since then $Aut(F_2) \sim \Psi_3$ (cf [8]).

Appendix A. Derivation of the Nielsen relations and proof of lemma 10

Since $R_1 \ldots R_{11}$ have previously been treated, it suffices to consider the remaining relations $R_{12} \ldots R_{20}$. Of these, R_{14} is trivial and R_{13}, R_{16} are not valid for $n = 3$. We turn to the remaining relations R_{12}, R_{15}, R_{17} and R_{18}.

R_{12}:

To deal with R_{12} we need an important intermediate result from relation Q_{20}. We multiply from the right with c_{23} and obtain for $c_{14}c_{23}$ the expression

$$\begin{aligned} c_{14}c_{23} &= \sigma_2 c_2 c_{14} c_2 \\ &= \sigma_2 \sigma_1 (c_{14})^{\sigma_1 c_2} \sigma_1 \\ &= (\sigma_1 \sigma_2 \sigma_3) P_{13}. \end{aligned} \tag{A1}$$

We used the relations of (24), (25) from B_3 in Nielsen form in the last step. Expression (A1) is remarkable since the left-hand side is an element of A_3, while the right-hand side is an element of B_3 in Nielsen form.

To construct the left-hand side of R_{12} we start from $c_{34} = c_{24}c_{23}c_{24}$ and obtain by use of the relations for B_3 in Nielsen form and by those indicated to the right

$$\begin{aligned}(c_{34})^{\sigma_1 c_2} &= P_{23}(c_{23})^{\sigma_1 c_2} P_{23}\\ &= P_{23}\sigma_1 c_2(c_{14}c_{23}c_2c_{23}c_{14}\sigma_2c_2)c_2\sigma_1 P_{23} \qquad (Q_{20})\\ &= P_{23}(\sigma_1\sigma_2\sigma_3)(\sigma_2c_2)P_{13}\sigma_2c_2P_{13}(\sigma_1\sigma_2\sigma_3)\sigma_2\sigma_1 P_{23} \qquad (Q_{19})\\ &= (\sigma_1\sigma_2\sigma_3)(P_{23}(\sigma_2c_2)P_{23})(P_{23}P_{13}\sigma_2c_2P_{13}P_{23})(\sigma_1\sigma_2)\\ &= (\sigma_1\sigma_2\sigma_3)(PQ^{-1}UQP)(Q^{-1}UQ)(\sigma_1\sigma_2).\end{aligned} \tag{A2}$$

In the last steps we used the expression $U = c_3\sigma_1c_3c_2 = \sigma_2c_2$ and equation (22).

To construct the right-hand side of R_{12} we use $c_{34} = c_{14}c_{23}c_{12}c_{23}c_{14}$ and derive from (20) and Φ_2, lemma 6,

$$c_{12} = c_3c_2c_3 = \sigma_2P_{12}\sigma_2c_2P_{12}\sigma_1\sigma_2. \tag{A3}$$

Inserting this expression and using (A1) we find

$$\begin{aligned}(c_{34})^{\sigma_1 c_2} &= \sigma_1c_2(\sigma_1\sigma_2\sigma_3)P_{13}(\sigma_2P_{12}\sigma_2c_2P_{12}\sigma_1)P_{13}(\sigma_1\sigma_2\sigma_3)c_2\sigma_1\\ &= (\sigma_1\sigma_2\sigma_3)\sigma_2c_2P_{13}P_{12}\sigma_2c_2P_{12}P_{13}c_2\sigma_2(\sigma_1\sigma_2) \qquad (Q_{19})\\ &= (\sigma_1\sigma_2\sigma_3)(UQ^{-1}UQU^{-1})(\sigma_1\sigma_2).\end{aligned} \tag{A4}$$

By equating the final expressions (A2) and (A4), and taking out the same factors on the left and on the right, we obtain relation R_{12}.

R_{15}:

This relation may be rewritten with (25) and σ_2 from (19) in the form

$$\sigma_2c_2 \rightleftharpoons \sigma_3. \tag{A5}$$

Since from (25) $\sigma_2 \rightleftharpoons \sigma_1$, $\sigma_{343} = \sigma_3 \rightleftharpoons \sigma_1$, R_{15} reduces to $c_2 \rightleftharpoons \sigma_{343}$, which is Q_{19}.

R_{17}:

To prove R_{17} we rewrite the left-hand side using (A1), by the steps

$$\begin{aligned}Q(\sigma_3c_{34})Q^{-1} &= P_{12}P_{13}(\sigma_3c_{34})P_{13}P_{12}\\ &= (\sigma_1\sigma_2\sigma_3)P_{12}c_{14}c_{23}(\sigma_3c_{34})c_{23}c_{14}P_{12}(\sigma_1\sigma_2\sigma_3)\\ &= \sigma_2c_2 = U\end{aligned} \tag{A6}$$

where commutators from Φ_2, lemma 6 were used in the last step. The right-hand side is transformed with lemma 6 as

$$P_{23}\sigma U\sigma P_{23} = Q(c_{12}\sigma_1)Q^{-1} \tag{A7}$$

so that R_{17} is equivalent to

$$R_{17}: \sigma_3c_{34} \rightleftharpoons c_{12}\sigma_1. \tag{A8}$$

This relation can be shown from relations in B_3 (7) and their conjugates, and from relations in A_3 (6).

R_{18}:

This relation in Φ_3 becomes

$$R_{18} : U \rightleftharpoons P_{13} U P_{13}. \tag{A9}$$

With the help of (A1), the right-hand side can be rewritten as

$$P_{13}\sigma_2 c_2 P_{13} = \sigma_3 q \sigma_2 \sigma_3. \tag{A10}$$

With Q_{19} we get the equivalent form

$$R_{18} : \sigma_2 c_2 \rightleftharpoons q\sigma_2 \,(Q_{19}, Q_{20}). \tag{A11}$$

In this form, the relation can be derived from Q_{20}, rewritten as

$$c_{23} = c_2\sigma_2 q = q\sigma_2 c_2. \tag{A12}$$

□

Note that relations Q_{19}, Q_{20}, not related to one of the subgroups Φ_2, A_3, B_3, are needed in the proof and hence are implicit in the Nielsen relations.

Appendix B. The normal subgroup $\langle T_1, T_2, T_3\rangle$; proof of proposition 13

The conjugation with the first four generators of $Aut(F_3)$ follows from proposition 12 and (34). It suffices to conjugate the three group elements with the remaining generator c_2. For this purpose we factorize the elements $S_i, i = 1, 2, 3$ into a first factor from the group $\Phi_2 = \langle c_2, c_3, \sigma_1\rangle$ of lemma 6 and a second factor with simple conjugation properties under Φ_2. These factorizations become

$$\begin{array}{lll} S_1 = v_1(X_1X_2)^2\sigma_{141} & v_1 = c_1\sigma_1\sigma_2c_1 & \sigma_{141} = (X_1X_2)^{-1}\sigma_{343}(X_1X_2) \\ S_2 = v_2(X_2)^2\sigma_{242} & v_2 = \sigma_1\sigma_2 & \sigma_{242} = (X_2)^{-1}\sigma_{343}X_2. \\ S_3 = v_3\sigma_{343} & v_3 = c_2\sigma_1\sigma_2c_2 & \end{array} \tag{B1}$$

Note that σ_{343} from relations Q_9, Q_{11}, Q_{19} commutes with all elements of Φ_2. Under conjugation with c_2 we get

$$\begin{array}{llll} & S_1 & S_2 & S_3 \\ (S_i)^{c_2}: & v_1X_1\sigma_{343}X_1 & v_3(X_2)^{-1}\sigma_{343}(X_2)^{-1} & v_2\sigma_{343}. \end{array} \tag{B2}$$

Next we derive for the automorphisms X_1, X_2 defined in (16) the relations

$$\sigma_{343}X_i\sigma_{343} \rightleftharpoons X_j \qquad ij = 11, 22, 12, 21. \tag{B3}$$

For $i = j$, equation (B3) follows from relation Q_{10}, written in terms of the generators of Φ_2, and for $ij = 21$ from conjugates of (A8). Using equations (B1)–(B3) and conjugations under Φ_2 one finds

$$\begin{array}{lll} (S_1S_2)^{c_2} & = & S_1S_3 = (S_1S_2)(S_2S_3) \\ (S_2S_3)^{c_2} & = & S_3S_2. \end{array} \tag{B4}$$

To conjugate S_3S_4 we make use of

$$S_3S_4 = c_{14}c_{23}S_2S_1c_{23}c_{14} \qquad c_2 = c_{14}c_{23}qc_{23}c_{14} \qquad q = \sigma_2c_2c_{23} \tag{B5}$$

to derive

$$(S_2S_1)^q = (S_2S_1) \leftrightarrow (S_3S_4)^{c_2} = S_3S_4. \tag{B6}$$

Equations (B4), (56) together with (34) yield the last row of (35) for $\langle T_1, T_2, T_3\rangle$ and lead to the normal property of the subgroup generated by them. □

References

[1] Garcia-Escudero J and Kramer P 1993 *Anales de Fisica, Monografias* **1**, vol 1 (Madrid) pp 339–42

[2] Garcia-Escudero J and Kramer P 1993 *J. Phys. A: Math. Gen.* **26** L1029–35

[3] Garcia-Escudero J and Kramer P 1993 *Proc. Int. Wigner Symposium* (Oxford)

[4] E J Humphreys 1990 *Reflection Groups and Coxeter Groups* (Cambridge: Cambridge University Press)

[5] Kramer P 1993 *Anales de Fisica, Monografias* **1**, vol 2 (Madrid) pp 370–3

[6] Kramer P 1993 *J. Phys. A: Math. Gen.* **26** 213–28

[7] Kramer P 1993 *J. Phys. A: Math. Gen.* **26** L245–50

[8] Kramer P 1994 *J. Phys. A: Math. Gen.* **27** 2011–22

[9] Magnus W, Karras A and Solitar D 1976 *Combinatorial Group Theory* (New York: Dover)

[10] Nielsen J 1924 *Math. Ann.* **91** 169–209